

EQUIVALENCE CLASSES OF POLARIZATIONS AND MODULI SPACES OF SHEAVES

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Introduction

Let X be a smooth algebraic variety over the complex number field \mathbb{C} with dimension n larger than one. For fixed c_1 in $\text{Pic}(X)$, c_2 in $A^2_{\text{num}}(X)$ which is the Chow group of codimension-two cycles on X modulo numerical equivalence and a polarization L on X , let $\mathcal{M}_L(c_1, c_2)$ be the moduli space of locally free rank-two sheaves stable with respect to L in the sense of Mumford-Takemoto such that their first and second Chern classes are c_1 and c_2 respectively. In this paper, we consider the problem: *what is the difference between $\mathcal{M}_{L_1}(c_1, c_2)$ and $\mathcal{M}_{L_2}(c_1, c_2)$ where L_1 and L_2 are two different polarizations?*

The understanding of this problem has two important implications. The first is in algebraic geometry. If one knows the structure of some moduli space $\mathcal{M}_L(c_1, c_2)$, then one will know the structure of any other moduli space $\mathcal{M}_{L'}(c_1, c_2)$ by comparing it with $\mathcal{M}_L(c_1, c_2)$. The author has applied this idea to the case where X is a ruled surface (for instance, see [21]); the results will appear elsewhere. The second implication is in gauge theory where X is an algebraic surface. When the geometric genus p_g of X is positive, the polynomials defined by Donaldson [6] are differential invariants. When p_g is zero, via the results in [4], these polynomials are defined on chambers of certain type (c_1, c_2) ; from the work of Mong [18] and Kotschick [15], one sees that we need to understand the difference between moduli spaces in order to compute these polynomials.

Our approach to the problem is to develop a theory about equivalence classes, walls and chambers of type (c_1, c_2) for polarizations on X . This is done in Chapter I. Fix c_1 and c_2 as before. Let L_1 and L_2 be two polarizations on X . We say that L_1 and L_2 are *equivalent* if every locally free rank-two sheaf V with first and second Chern classes c_1 and c_2 , respectively, is L_1 -stable if and only if it is L_2 -stable.

Theorem 1. *Let V be a locally free rank-two sheaf which is L_1 -stable and L_2 -unstable. Then, there is an invertible sheaf $\mathcal{O}_X(F)$, and integers i, j with $0 \leq i \leq j \leq (n - 1)$ such that, S being equal to $L_1^{n-1-j} \cdot L_2^i \cdot (L_1 + L_2)^{j-1-i}$, we have*

- (i) $\mathcal{O}_X(F + c_1(V))$ is divisible by 2 in $\text{Pic}(X)$;
- (ii) $[c_1(V)^2 - 4c_2(V)] \cdot S \leq F^2 \cdot S < 0$;
- (iii) $(F \cdot L_1) \cdot S < 0 < (F \cdot L_2) \cdot S$.

The special case where X is a surface and c_1 is trivial (see Proposition 1.2.1 in Chapter I) is obtained by Mong [19] and Friedman [7]. Using Theorem 1, we can define walls and chambers of type (c_1, c_2) . Let $\text{Num}(X)$ be the group of divisors on X modulo numerical equivalence relation. Let C_X be the Kähler cone in $\text{Num}(X) \otimes \mathbf{R}$ generated by all ample divisors. Roughly speaking, a wall of type (c_1, c_2) is the intersection of C_X with a set of the form $\{x \in \text{Num}(X) \otimes \mathbf{R} \mid x \cdot \zeta \cdot S = 0\}$ where ζ and S satisfy some conditions. Walls of type (c_1, c_2) cut C_X into many connected components; each of these components is a chamber of type (c_1, c_2) , and intersection of a chamber with $\text{Num}(X)$ is \mathbf{Z} -chamber. A basic relation among equivalence classes, walls and chambers is that an equivalence class is a union of \mathbf{Z} -chambers, and possibly some polarizations lying on walls. We will see that the study of moduli spaces of locally free rank-two sheaves stable with respect to polarizations lying on walls can be reduced to the study of moduli spaces of locally free rank-two sheaves stable with respect to polarizations lying in chambers.

In Chapter II, we focus on the case where X is an algebraic surface, and work in the Kähler cone C_X . It turns out that the concepts of walls and chambers here are slight modifications of those used by Donaldson [5] and Friedman and Morgan [8]. Let ζ be any numerical equivalence class which defines a nonempty wall W^ζ of type (c_1, c_2) . We introduce the notation $E_\zeta(c_1, c_2)$, which is the set of all locally free rank-two sheaves V given by nontrivial extensions

$$0 \rightarrow \mathcal{O}_X(F) \rightarrow V \rightarrow \mathcal{O}_X(c_1 - F) \otimes I_Z \rightarrow 0,$$

where F is some divisor with $(2F - c_1) \equiv \zeta$, and Z is some locally complete intersection 0-cycle with length $l(Z) = c_2 + (\zeta^2 - c_1^2)/4$. Every sheaf V in $E_\zeta(c_1, c_2)$ has two basic properties: (a) the above defining exact sequence for V is canonical; (b) V is L -stable if $L \cdot \zeta$ is negative and L is contained in a chamber \mathcal{C} such that $W^\zeta \cap \text{Closure}(\mathcal{C})$ is nonempty. Our main result is the following.

Theorem 2. Let \mathcal{E} , \mathcal{E}_1 be two chambers sharing a common wall. Then, as sets,

$$\mathcal{M}_{\mathcal{E}}(c_1, c_2) = \left(\mathcal{M}_{\mathcal{E}_1}(c_1, c_2) - \coprod E_{(-\zeta)}(c_1, c_2) \right) \coprod \left(\coprod E_{\zeta}(c_1, c_2) \right),$$

where ζ satisfies $\zeta \cdot L < 0$ for some $L \in \mathcal{E}$, and runs over all numerical equivalence classes which define the common wall.

We now explain the geometric meaning of Theorem 2. It is well known that $\mathcal{M}_L(c_1, c_2)$ is quasiprojective (see [16]). Thus, for two polarizations L_1 and L_2 , one would expect to perform only a finite surgery operation to pass from $\mathcal{M}_{L_1}(c_1, c_2)$ to $\mathcal{M}_{L_2}(c_1, c_2)$ by removing and adding locally closed subsets. Indeed, $E_{\zeta}(c_1, c_2)$ is a finite disjoint union of locally closed subsets of $\mathcal{M}_{\mathcal{E}}(c_1, c_2)$ under the conditions in Theorem 2. Therefore, Theorem 2 gives a precise description about what locally closed subsets should be removed or added when L_1 and L_2 can be connected by a path which intersects with only one wall of type (c_1, c_2) .

We notice that $E_{\zeta}(c_1, c_2)$ plays a significant role in comparing different moduli spaces and in determining the equivalence classes of polarizations. In §2 of Chapter II, we discuss the nonemptiness of $E_{\zeta}(c_1, c_2)$. We use some classical techniques to construct locally free rank-two sheaves. In this direction, we have two results.

Theorem 3. $E_{\zeta}(c_1, c_2)$ or $E_{(-\zeta)}(c_1, c_2)$ is nonempty if $c_2 > c(X, c_1)$ where $c(X, c_1)$ is a constant depending on X and c_1 .

Notice that no condition is imposed on the wall W^{ζ} . If we do make some assumption on W^{ζ} , we obtain a stronger result. The second is the following.

Theorem 4. Let \mathcal{B} be a compact subset in the Kähler cone C_X . For any numerical equivalence class ζ with $W^{\zeta} \cap \mathcal{B}$ to be nonempty, $E_{\zeta}(c_1, c_2)$ is nonempty when $c_2 > c(X, c_1, \mathcal{B})$ where $c(X, c_1, \mathcal{B})$ is a constant depending on X , c_1 and \mathcal{B} .

In [23], we have estimated the dimension of $E_{\zeta}(c_1, c_2)$ and studied the birational structure of $\mathcal{M}_L(c_1, c_2)$ when $(4c_2 - c_1^2)$ is sufficiently large.

CHAPTER I

1. Stability under different polarizations

1.1. Stability in the sense of Mumford-Takemoto. We begin with several definitions. Let X be a smooth projective variety of dimension n over the complex number field \mathbb{C} . From the exponential sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X^* \rightarrow 0,$$

we obtain a map from $\text{Pic}(X) \rightarrow H^2(X; \mathbb{Z})$. Let $\text{Num}(X)$ be a $\text{Pic}(X)$ modulo the kernel of the induced map $\text{Pic}(X) \rightarrow H^2(X; \mathbb{Z})/\text{Torsion}$. Clearly, $\text{Num}(X)$ is a finely generated free abelian group. The images of ample invertible sheaves in $\text{Num}(X)$ are called *polarizations* (a word of caution: this definition is different from the standard one). Since we will only need numerical properties, we make no distinctions between a polarization and its inverse images in $\text{Pic}(X)$ or $\text{Div}(X)$.

Definition 1.1.1. For a polarization L and a torsion free coherent sheaf V , let

$$\mu_L(V) = \frac{1}{\text{rank}(V)}(c_1(V) \cdot L^{n-1}).$$

Definition 1.1.2. Let V be a torsion free coherent sheaf on X . V is L -stable (resp. L -semistable) if, for all coherent subsheaves W of V with $0 < \text{rank}(W) < \text{rank}(V)$, we have $\mu_L(W) < \mu_L(V)$ (resp. $\mu_L(W) \leq \mu_L(V)$). V is said to be *unstable* if it is not semistable, and *strictly semistable* if it is semistable but not stable.

Remark 1.1.3. If $n = 2$ or $\text{rank}(V) = 2$, then it is sufficient to check (semi)stability on locally free subsheaves of V such that the quotients are torsion free.

Definition 1.1.4. Fix $c_1 \in \text{Pic}(X)$ and $c_2 \in A_{\text{num}}^2(X)$ where $A_{\text{num}}^i(X)$ is the Chow group of cycles on X of codimension i modulo numerical equivalence. Let L be a polarization on X . We define $\mathcal{M}_L(c_1, c_2)$ to be the moduli space of L -stable locally free rank-two sheaves with fixed c_1 and c_2 .

Remark 1.1.5. It is well known that $\mathcal{M}_L(c_1, c_2)$ is quasiprojective (see [16]).

1.2. Stability under different polarizations.

Proposition 1.2.1 (see [19] and [7]). *Let V be a locally free rank-two sheaf on a smooth algebraic surface X with $c_1(V) = 0$. Let L_1 and L_2 be two polarizations on X . Suppose that V is L_1 -stable and L_2 -unstable. Then, there exists an invertible sheaf $\mathcal{O}_X(F)$ on X with $L_1 \cdot F < 0 < L_2 \cdot F$ and $-c_2(V) \leq F^2 < 0$.*

A slight modification of the proof in [7] of the proposition above will result in the following which says that $c_1(V) \neq 0$ makes no big difference.

Proposition 1.2.2. *Let V be a locally free rank-two sheaf on a smooth surface X . Let L_1 and L_2 be two polarizations on X . Suppose that V is L_1 -stable and L_2 -unstable. Then, there exists an invertible sheaf $\mathcal{O}_X(F)$ such that $\mathcal{O}_X(F + c_1(V))$ is divisible by 2 in $\text{Pic}(X)$, $L_1 \cdot F < 0 < L_2 \cdot F$ and $c_1(V)^2 - 4c_2(V) \leq F^2 < 0$.*

Corollary 1.2.3. *Let V be a locally free rank-two sheaf on a smooth surface X . If V is stable with respect to one polarization and unstable with respect to another polarization, then $c_1(V)^2 - 4c_2(V) < 0$.*

Remark 1.2.4. This corollary is obtained by Takemoto [25]. It is a special case of Bogomolov's instability theorem [1]: if $c_1^2 > 4c_2$, then $\mathcal{M}_L(c_1, c_2) = \emptyset$ for any polarization L on any surface X . Donaldson [4] and Kobayashi [13] (see also [17] and [14]) showed that if $c_1(V) = c_2(V) = 0$, then V is stable with respect to some polarization on X if and only if V comes from an irreducible unitary representation of the fundamental group $\pi_1(X)$. In general, the case $(4c_2 - c_1^2) = 0$ has been studied by Takemoto [24, 25]. Therefore, in case of algebraic surfaces, we always assume $(4c_2 - c_1^2) > 0$ unless otherwise specified.

Theorem 1.2.5. *Let V be a locally free rank-two sheaf on a smooth n -dimensional variety X where $n \geq 2$. Let L_1 and L_2 be two polarizations on X . Suppose that V is L_1 -stable and L_2 -unstable. Then, there exist an invertible sheaf $\mathcal{O}_X(F)$ on X , and integers i, j satisfying $0 \leq i < j \leq (n-1)$ such that, S being equal to $L_1^{n-1-j} \cdot L_2^i \cdot (L_1 + L_2)^{j-i-1}$, we have*

- (i) $\mathcal{O}_X(F + c_1(V))$ is divisible by 2 in $\text{Pic}(X)$;
- (ii) $[c_1(V)^2 - 4c_2(V)] \cdot S \leq F^2 \cdot S < 0$;
- (iii) $(F \cdot L_1) \cdot S < 0 < (F \cdot L_2) \cdot S$.

Proof. By assumption, there exists an invertible subsheaf $\mathcal{O}_X(G)$ of V such that $G \cdot L_1^{n-1} < [c_1(V) \cdot L_1^{n-1}]/2$ and $[c_1(V) \cdot L_2^{n-1}]/2 < G \cdot L_2^{n-1}$. By Remark 1.1.3, we may assume that the quotient $V/\mathcal{O}_X(G)$ is torsion free. Put $F = 2G - c_1(V)$. We have $F \cdot L_1^{n-1} < 0 < F \cdot L_2^{n-1}$. Thus, we can choose integers i and j with $0 \leq i < j \leq (n-1)$ such that

- (i) $F \cdot L^{(n-1-k)} \cdot L_2^k \leq 0$ for $k < i$;
- (ii) $F \cdot L^{(n-1-k)} \cdot L_2^k < 0$ for $k = i$;
- (iii) $F \cdot L^{(n-1-k)} \cdot L_2^k = 0$ for $i < k < j$;
- (iv) $F \cdot L^{(n-1-k)} \cdot L_2^k > 0$ for $k = j$.

Put $S = L_1^{n-1-j} \cdot L_2^i \cdot (L_1 + L_2)^{j-i-1}$. Then, we have

$$(F \cdot L_1) \cdot S = F \cdot L_1^{n-1-i} \cdot L_2^i < 0 \quad \text{and} \quad (F \cdot L_2) \cdot S = F \cdot L_1^{(n-1-j)} \cdot L_2^j > 0.$$

Therefore, we obtain that $(F \cdot L_1) \cdot S < 0 < (F \cdot L_2) \cdot S$. Note that after a scaling of L_1 and L_2 , we may regard S as a smooth algebraic surface in X . Restrict F , L_1 and L_2 to S . Put $H = (F|_S \cdot L_2|_S) \cdot L_1|_S - (F|_S \cdot L_1|_S) \cdot L_2|_S$. Then H is ample on S and $F|_S \cdot H = 0$. By the Hodge Index Theorem [11], either $(F|_S)^2 < 0$ or $F|_S \equiv 0$. But the second case

will not happen since $F|_S \cdot L_1|_S < 0$. Thus, $(F|_S)^2 < 0$, i.e., $F^2 \cdot S < 0$. On the other hand, there exists an exact sequence

$$0 \rightarrow \mathcal{O}_X(G) \rightarrow V \rightarrow \mathcal{O}_X(c_1(V) - G) \otimes I_Z \rightarrow 0,$$

where Z is a locally complete intersection codimension-two cycle on X . Thus, $c_2(V) = G \cdot (c_1(V) - G) + [Z]$, and $F^2 \cdot S = [c_1(V)^2 - 4c_2(V)] \cdot S + 4[Z] \cdot S \geq [c_1(V)^2 - 4c_2(V)] \cdot S$. Hence, we have $[c_1(V)^2 - 4c_2(V)] \cdot S \leq F^2 \cdot S < 0$.

2. Equivalence classes and chambers

2.1. Equivalence classes of polarizations.

Definition 2.1.1. Let L_1, L_2 be two polarizations on X . Fix $c_1 \in \text{Pic}(X)$ and $c_2 \in A_{\text{num}}^2(X)$. We define $L_1 \overset{s}{\geq} L_2$ if every locally free rank-two sheaf with c_1 and c_2 as its first and second Chern classes is L_1 -stable whenever it is L_2 -stable. We define $L_1 \overset{s}{=} L_2$ if both $L_1 \overset{s}{\geq} L_2$ and $L_2 \overset{s}{\geq} L_1$.

Remark 2.1.2. Fix c_1 and c_2 . Then $L_1 \overset{s}{=} L_2$ means that the moduli spaces $\mathcal{M}_{L_1}(c_1, c_2)$ and $\mathcal{M}_{L_2}(c_1, c_2)$ can be naturally identified.

Notation 2.1.3. For a polarization L , we put

- (i) $\Delta_L = \{L' \mid L' \text{ is a polarization and } L' \overset{s}{\geq} L\}$;
- (ii) $\mathcal{E}_L = \{L' \mid L' \text{ is a polarization and } L' \overset{s}{=} L\}$.

Proposition 2.1.4. Let L and L' be two polarizations on X . Then,

- (i) $\Delta_{L'} \subseteq \Delta_L$ if and only if $L \overset{s}{\leq} L'$;
- (ii) $\Delta_{L'} = \Delta_L$ if and only if $L \overset{s}{=} L'$.

Proof. Follows from Definition 2.1.1 and Notation 2.1.3. q.e.d.

We already know that $\text{Num}(X)$ is a finitely generated free abelian group. There is an open cone (called the *Kähler cone*) C_X , in $\text{Num}(X) \otimes \mathbf{R}$ which is spanned by polarizations. Fix $c_1 \in \text{Pic}(X)$ and $c_2 \in A_{\text{num}}^2(X)$.

Definition 2.1.5. (i) Let $S \in A_{\text{num}}^{n-2}(X)$, and $\zeta \in \text{Num}(X) \otimes \mathbf{R}$. We define

$$W^{(\zeta, S)} = C_X \cap \{x \in \text{Num}(X) \otimes \mathbf{R} \mid x \cdot \zeta \cdot S = 0\}.$$

(ii) We define $\mathcal{W}(c_1, c_2)$ to be the set whose elements consist of $W^{(\zeta, S)}$, where S is a complete intersection surface in X , and ζ is the numerical equivalence class of a divisor F on X such that $\mathcal{O}_X(F + c_1)$ is divisible by 2 in $\text{Pic}(X)$, and that

$$F^2 \cdot S < 0, \quad c_2 + \frac{F^2 - c_1^2}{4} = [Z]$$

for some locally complete intersection codimension-two cycle Z in X .

(iii) A wall of type (c_1, c_2) is an element in $\mathcal{W}(c_1, c_2)$. A chamber of type (c_1, c_2) is a connected component of $C_X - \mathcal{W}(c_1, c_2)$. A \mathbf{Z} -chamber of type (c_1, c_2) is the intersection of $\text{Num}(X)$ with some chamber of type (c_1, c_2) .

We notice that when $\dim(X) = 2$, the above definitions for walls and chambers are slight modifications of those used in [5], [8], [18] and [15]; they also appeared briefly in [7]. As in §1, Chapter II of [8], we can show the following result.

Proposition 2.1.6. *The set of walls of type (c_1, c_2) is locally finite if $\dim(X) = 2$.*

Remark 2.1.7. If $\dim(X) > 2$, Proposition 2.1.6 will not hold anymore (see 2.3 below). This will limit the application of our theory in higher-dimensional cases. Nevertheless, this theory is quite satisfactory in dimension two (see Chapter II).

2.2. Some relations among walls, chambers and equivalence classes.

Proposition 2.2.1. *Let \mathcal{C} be a chamber, and let $L_1, L_2 \in \mathcal{C}$. Then,*

$$L_1 \stackrel{s}{\leq} L_2 \stackrel{s}{\leq} (L_1 + L_2).$$

Proof. By assumption, L_1 and L_2 are not separated by any wall. Thus, by the Theorem 1.2.5 in §1, $L_1 \stackrel{s}{\leq} L_2$. Since \mathcal{C} is convex and closed under the action of \mathbf{R}^+ , $(L_1 + L_2)$ is also in \mathcal{C} .

Corollary 2.2.2. *Each \mathbf{Z} -chamber is contained in some equivalence class. Thus, an equivalence class is a union of \mathbf{Z} -chambers, and possibly some polarizations lying on walls.*

In the next chapter, we will discuss this corollary for algebraic surfaces in detail. Right now, using Corollary 2.2.2, we can make

Definition 2.2.3. Fix $c_1 \in \text{Pic}(X)$ and $c_2 \in A_{\text{num}}^2(X)$. Let \mathcal{C}_1 and \mathcal{C}_2 be two \mathbf{Z} -chambers of type (c_1, c_2) , and $L_1 \in \mathcal{C}_1$, $L_2 \in \mathcal{C}_2$.

- (i) We define that $\mathcal{C}_1 \stackrel{s}{\geq} \mathcal{C}_2$ if $L_1 \stackrel{s}{\geq} L_2$, and that $\mathcal{C}_1 \stackrel{s}{=} \mathcal{C}_2$ if $L_1 \stackrel{s}{=} L_2$.
- (ii) We define that a locally free rank-two sheaf is \mathcal{C}_1 -stable if it is L_1 -stable. Let $\mathcal{M}_{\mathcal{C}_1}(c_1, c_2)$ be $\mathcal{M}_{L_1}(c_1, c_2)$.
- (iii) We define $\Delta_{\mathcal{C}_1}$ to be Δ_{L_1} , and $\mathcal{E}_{\mathcal{C}_1}$ to be \mathcal{E}_{L_1} .

Proposition 2.2.4. *Let \mathcal{C} and \mathcal{C}' be two chambers having a unique common face which is part of the wall W . Assume that the two \mathbf{Z} -chambers $\text{Num}(X) \cap \mathcal{C}$ and $\text{Num}(X) \cap \mathcal{C}'$ are nonempty. If W is not of the form $W^{(F, S)}$ where S is a smooth complete intersection surface in X and $F = 2G - c_1$ for some invertible subsheaf $\mathcal{O}_X(G)$ of a locally free rank-two sheaf*

which is either \mathcal{E} -stable or \mathcal{E}' -stable, then

$$\text{Num}(X) \cap \mathcal{E} \stackrel{s}{=} \text{Num}(X) \cap \mathcal{E}' .$$

Moreover, if $\dim(X) = 2$, the converse is also true.

Proof. The first statement follows from Definition 2.1.1 and Theorem 1.2.5.

Next, suppose $\dim(X) = 2$ and $\text{Num}(X) \cap \mathcal{E} \stackrel{s}{=} \text{Num}(X) \cap \mathcal{E}'$. If $W = W^F$ where $\mathcal{O}_X(F)$ is a subsheaf of V which is $(\text{Num}(X) \cap \mathcal{E})$ -stable (so must be $(\text{Num}(X) \cap \mathcal{E}')$ -stable), then by the definition of stability, $F \cdot L < 0$ for any $L \in \text{Num}(X) \cap \mathcal{E}$ or $\text{Num}(X) \cap \mathcal{E}'$. Thus, W cannot separate \mathcal{E} and \mathcal{E}' . A contradiction. q.e.d.

In the following, we consider polarizations lying on walls.

Proposition 2.2.5. *Suppose \mathcal{E}_1 and \mathcal{E}_2 are two chambers having a unique common face which is part of the wall W . Let $\mathcal{F} = W \cap \text{Closure}(\mathcal{E}_i)$ be the common face. Assume that $\text{Num}(X) \cap \mathcal{F}$ is nonempty and $L \in \text{Num}(X) \cap \mathcal{F}$. Then,*

- (i) $\text{Num}(X) \cap \mathcal{F}$ is contained in one equivalence class;
- (ii) $\Delta_L \supseteq (\text{Num}(X) \cap \mathcal{E}_1) \cap (\text{Num}(X) \cap \mathcal{E}_2)$;
- (iii) $\text{Num}(X) \cap \mathcal{E}_1 \stackrel{s}{=} \text{Num}(X) \cap \mathcal{E}_2$ if and only if $\mathcal{E}_L \supseteq (\text{Num}(X) \cap \mathcal{E}_1) \cup (\text{Num}(X) \cap \mathcal{E}_2)$.

Proof. (i) Follows from the fact that no wall intersects with \mathcal{F} properly.

(ii) We need only to show that $\Delta_L \supseteq (\text{Num}(X) \cap \mathcal{E}_1)$. Assume $\text{Num}(X) \cap \mathcal{E}_1$ is nonempty, and let $L_1 \in \text{Num}(X) \cap \mathcal{E}_1$. Suppose V is L -stable but is not $(\text{Num}(X) \cap \mathcal{E}_1)$ -stable. Then there exists a wall $W' = W^{(F,S)}$ such that $F \cdot L \cdot S < 0 \leq F \cdot L_1 \cdot S$ by the Theorem 1.2.5 in §1. This implies that W' separates L and L_1 , and does not contain L . But the only wall separating L and L_1 is W which contains L . A contradiction. Thus, V is $(\text{Num}(X) \cap \mathcal{E}_1)$ -stable and $\Delta_L \supseteq (\text{Num}(X) \cap \mathcal{E}_1)$.

(iii) Clearly, if $\mathcal{E}_L \supseteq (\text{Num}(X) \cap \mathcal{E}_1) \cup (\text{Num}(X) \cap \mathcal{E}_2)$, then $\text{Num}(X) \cap \mathcal{E}_1 \stackrel{s}{=} \text{Num}(X) \cap \mathcal{E}_2$. Suppose $\text{Num}(X) \cap \mathcal{E}_1 \stackrel{s}{=} \text{Num}(X) \cap \mathcal{E}_2$, and V is $(\text{Num}(X) \cap \mathcal{E}_i)$ -stable ($i = 1, 2$). If V is not L -stable, then there is a wall $W' = W^{(F,S)}$ such that $F \cdot L \cdot S \geq 0 > F \cdot \mathcal{E}_i \cdot S$. Thus, $W = W'$ and W cannot separate \mathcal{E}_1 and \mathcal{E}_2 . Contradiction! Therefore, V must be L -stable, so $L \stackrel{s}{\geq} \text{Num}(X) \cap \mathcal{E}_i$. Combining with (ii), we conclude that $L \stackrel{s}{=} (\text{Num}(X) \cap \mathcal{E}_1)$. So $\mathcal{E}_L \supseteq (\text{Num}(X) \cap \mathcal{E}_1) \cup (\text{Num}(X) \cap \mathcal{E}_2)$.

Remark 2.2.6. From the proof, we see that V is $(\text{Num}(X) \cap \mathcal{F})$ -stable if and only if it is both $(\text{Num}(X) \cap \mathcal{E}_1)$ -stable and $(\text{Num}(X) \cap \mathcal{E}_2)$ -stable. Thus, the study of moduli spaces of locally free rank-two sheaves stable

with respect to polarizations lying on walls may be reduced to the study of moduli spaces of locally free rank-two sheaves with respect to polarizations lying in \mathbf{Z} -chambers.

2.3. An example. Let $X = \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$, and let p_i be the i -th projection of X to \mathbf{P}^1 . Put $\mathcal{L}_i = p_i^* \mathcal{O}_{\mathbf{P}^1}(1)$. Let (s_1, s_2, s_3) denote $(s_1 \cdot \mathcal{L}_1 + s_2 \cdot \mathcal{L}_2 + s_3 \cdot \mathcal{L}_3)$. Then, (s_1, s_2, s_3) is a polarization if and only if s_1, s_2, s_3 are positive. The proof of the following is quite elementary (see [21]); hence we simply state the result.

Proposition 2.3.1. *Let $c_1 = 0$ and $c_2 = 2(\mathcal{L}_2 \cdot \mathcal{L}_3 + \mathcal{L}_1 \cdot \mathcal{L}_3)$. Then,*

- (i) *there is no \mathbf{Z} -chamber of type $(0, c_2)$;*
- (ii) *chambers of type $(0, c_2)$ exist, and each chamber consists of a ray;*
- (iii) *$(1, 1, m)$ is equivalent to $(1, 1, n)$ if $m, n > 1$.*

CHAPTER II

1. Theory in the case of algebraic surfaces

1.1. Remarks on polarizations. One essential difference between Proposition 1.2.2 and Theorem 1.2.5 in Chapter I is that when $\dim(X) > 2$, we have restrictions on the complete intersection surface S in X , and we have to work with those elements in $\text{Num}(X) \cap \mathbf{C}_X$ in order to guarantee that the S in Theorem 1.2.5 of Chapter I is indeed an algebraic surface up to a scaling. But in the case where X is a surface, S disappears. This implies that we can work with any element in \mathbf{C}_X .

Definition 1.1.1. A polarization on an algebraic surface X is an element in \mathbf{C}_X .

All definitions and conclusions in Chapter I are true if we simply replace \mathbf{Z} -chambers and polarizations by chambers and generalized polarizations above.

1.2. Introduction of $E_\zeta(c_1, c_2)$. From now on, X is an algebraic surface. Fix $c_1 \in \text{Pic}(X)$ and $c_2 \in \mathbf{Z}$ such that $(4c_2 - c_1^2) > 0$. Let L_1 and L_2 be two polarizations which are not equivalent. We may assume that there is a locally free rank-two sheaf V which is L_1 -stable but not L_2 -stable. From the proof of Theorem 1.2.5 in Chapter I, V sits in

$$0 \rightarrow \mathcal{O}_X(G) \rightarrow V \rightarrow \mathcal{O}_X(c_1 - G) \otimes I_Z \rightarrow 0,$$

where Z is a locally complete intersection 0-cycle on X , and $(2G - c_1)$ defines a nonempty wall of type (c_1, c_2) with $L_1 \cdot (2G - c_1) < 0 \leq L_2 \cdot (2G - c_1)$. Notice that the extension above is nontrivial since a stable sheaf cannot be a direct sum.

We want to study the inverse process. First of all, we introduce $E_\zeta(c_1, c_2)$.

Definition 1.2.1. Let ζ be some numerical equivalence class which defines a nonempty wall of type (c_1, c_2) . We define $E_\zeta(c_1, c_2)$ to be the set of all locally free rank-two sheaves V given by nontrivial extensions of $\mathcal{O}_X(c_1 - F) \otimes I_Z$ by $\mathcal{O}_X(F)$

$$0 \rightarrow \mathcal{O}_X(F) \rightarrow V \rightarrow \mathcal{O}_X(c_1 - F) \otimes I_Z \rightarrow 0,$$

where F is some divisor with $(2F - c_1) \equiv \zeta$, and Z is some locally complete intersection 0-cycle with length $l(Z) = c_2 + (\zeta^2 - c_1^2)/4$.

The following lemma implies that the above extension is canonical.

Lemma 1.2.2. Let V be a sheaf in $E_\zeta(c_1, c_2)$, and let $\mathcal{O}_X(F)$ be its subsheaf coming from the extension in Definition 1.2.1. Then, $\text{Hom}(\mathcal{O}_X(F), V) \cong \mathbf{C}$; moreover, $\mathcal{O}_X(F_1)$ is equal to $\mathcal{O}_X(F)$ if $\mathcal{O}_X(F_1)$ is a subsheaf of V with $(2F_1 - c_1) \equiv \zeta$.

Proof. Note that $(c_1 - F - F_1) \equiv -\zeta$. Since $\zeta \cdot L > 0$ for some polarization L , neither $(c_1 - 2F)$ nor $(c_1 - F - F_1)$ can be effective. Let $L_0 \in W^\zeta$. Then, V is strictly L_0 -semistable. Thus, $V/\mathcal{O}_X(F_1)$ is torsion free. Therefore, the conclusions follow from the standard fact: two invertible subsheaves of a sheaf with torsion free quotients coincide if the map of one to the quotient by the other is zero.

Theorem 1.2.3. Assume ζ defines a nonempty wall W of type (c_1, c_2) . Let \mathcal{C} be a chamber such that $W \cap \text{Closure}(\mathcal{C})$ is nonempty and $L \cdot \zeta < 0$ for $L \in \mathcal{C}$. Let $L_0 \in W$, and let L_1 be a polarization with $L_1 \cdot \zeta > 0$. If V is contained in $E_\zeta(c_1, c_2)$, then V is L_1 -unstable, strictly L_0 -semistable and L -stable.

Proof. Obviously, V is L_1 -unstable and strictly L_0 -semistable. In the following, we show that V is L -stable. By definition, V sits in a nontrivial extension

$$0 \rightarrow \mathcal{O}_X(F) \rightarrow V \rightarrow \mathcal{O}_X(c_1 - F) \otimes I_Z \rightarrow 0,$$

where F is some divisor with $(2F - c_1) \equiv \zeta$. Let $\mathcal{O}_X(F_1)$ be any invertible subsheaf of V with torsion free quotient. Then, we have either

$$0 \rightarrow \mathcal{O}_X(F_1) \rightarrow \mathcal{O}_X(F) \quad \text{or} \quad 0 \rightarrow \mathcal{O}_X(F_1) \rightarrow \mathcal{O}_X(c_1 - F) \otimes I_Z.$$

If $0 \rightarrow \mathcal{O}_X(F_1) \rightarrow \mathcal{O}_X(F)$, then $L \cdot F_1 < (L \cdot c_1)/2$. Assume $0 \rightarrow \mathcal{O}_X(F_1) \rightarrow \mathcal{O}_X(c_1 - F) \otimes I_Z$. Then, $(c_1 - F - F_1)$ is strictly effective. Thus, $L_0 \cdot (2F_1 - c_1) < 0$. We claim that $L \cdot F_1 < (L \cdot c_1)/2$: if $L \cdot F_1 \geq (L \cdot c_1)/2$, then

$$L'_0 \cdot (2F_1 - c_1) < 0 \leq L \cdot (2F_1 - c_1),$$

where we choose $L'_0 \in W \cap \text{Closure}(\mathcal{E})$; on the other hand, V sits in an extension

$$0 \rightarrow \mathcal{O}_X(F_1) \rightarrow V \rightarrow \mathcal{O}_X(c_1 - F_1) \otimes I_{Z_1} \rightarrow 0;$$

as in the proof of Theorem 1.2.5 in Chapter I, $-(4c_2 - c_1^2) \leq (2F_1 - c_1)^2 < 0$; combined with $L'_0 \cdot (2F_1 - c_1) < 0 \leq L \cdot (2F_1 - c_1)$, this implies that $(2F_1 - c_1)$ defines a nonempty wall which separates L'_0 and L , and does not contain L'_0 ; but $L \in \mathcal{E}$ and $L'_0 \in W \cap \text{Closure}(\mathcal{E})$, so any wall which separates L and L'_0 must contain L'_0 ; we thus obtain a contradiction. In any case, $L \cdot F_1 < (L \cdot c_1)/2$, so V is L -stable.

Corollary 1.2.4. *Let \mathcal{E} be a chamber such that $W^\zeta \cap \text{Closure}(\mathcal{E})$ is nonempty and $L \cdot \zeta < 0$ for $L \in \mathcal{E}$. Then, $E_\zeta(c_1, c_2)$ is embedded in $\mathcal{M}_\mathcal{E}(c_1, c_2)$.*

Proof. By Theorem 1.2.3, each sheaf in $E_\zeta(c_1, c_2)$ is stable with respect to \mathcal{E} . By Lemma 1.2.2, $E_\zeta(c_1, c_2)$ is embedded in $\mathcal{M}_\mathcal{E}(c_1, c_2)$.

Proposition 1.2.5. *Let ζ and η be two different numerical equivalence classes defining nonempty walls of type (c_1, c_2) . Then, $E_\zeta(c_1, c_2)$ and $E_\eta(c_1, c_2)$ have no intersection if either of the following is true:*

- (i) both $\zeta \cdot L'$ and $\eta \cdot L'$ are nonnegative for some polarization L' ;
- (ii) the two walls W^ζ and W^η are coincident.

Proof. We may assume that both $E_\zeta(c_1, c_2)$ and $E_\eta(c_1, c_2)$ are nonempty. Suppose $V \in E_\zeta(c_1, c_2) \cap E_\eta(c_1, c_2)$. Let \mathcal{E} be a chamber such that one of its faces is contained in W^ζ . Without loss of generality, let $L \cdot \zeta < 0$ for some $L \in \mathcal{E}$. By Theorem 1.2.3, V is L -stable. Since $V \in E_\eta(c_1, c_2)$, $L \cdot \eta < 0$ by Definition 1.2.1.

- (i) By definition, V fits into two extensions

$$\begin{aligned} 0 \rightarrow \mathcal{O}_X(F) \rightarrow V \rightarrow \mathcal{O}_X(c_1 - F) \otimes I_Z \rightarrow 0, \\ 0 \rightarrow \mathcal{O}_X(G) \rightarrow V \rightarrow \mathcal{O}_X(c_1 - G) \otimes I_U \rightarrow 0, \end{aligned}$$

where $(2F - c_1) \equiv \zeta$ and $(2G - c_1) \equiv \eta$. Note that $(c_1 - F - G) \equiv -(\zeta + \eta)/2$. Since $L \cdot (\zeta + \eta) < 0$, $(c_1 - F - G)$ cannot be trivial; since $L' \cdot (\zeta + \eta) \geq 0$, $(c_1 - F - G)$ cannot be effective. Applying the standard fact in the proof of Lemma 1.2.2, we conclude that $\mathcal{O}_X(F) = \mathcal{O}_X(G)$, so $\zeta = \eta$. A contradiction.

- (ii) Since $L \cdot \eta < 0$ and $\eta = a\zeta$ for some number a , $a > 0$. Choose a polarization L' with $\zeta \cdot L' > 0$, and our conclusion follows from (i).

1.3. Set-theoretic comparison of moduli spaces.

Proposition 1.3.1. *Let \mathcal{C} be a chamber, and \mathcal{F} be one of its faces. Then, as sets,*

$$\mathcal{M}_{\mathcal{C}}(c_1, c_2) = \mathcal{M}_{\mathcal{F}}(c_1, c_2) \coprod \left(\coprod_{\zeta} E_{\zeta}(c_1, c_2) \right),$$

where ζ satisfies $\zeta \cdot L < 0$ for some $L \in \mathcal{C}$, and runs over all numerical equivalence classes which define the wall containing \mathcal{F} .

Proof. First of all, by Theorem 1.2.3 and Proposition 1.2.5 (ii), the right-hand side consists of disjoint unions. Next, by Remark 2.2.6 in Chapter I (replace \mathbf{Z} -chambers by chambers), $\mathcal{M}_{\mathcal{F}}(c_1, c_2)$ is contained in $\mathcal{M}_{\mathcal{C}}(c_1, c_2)$; by Corollary 1.2.4, $E_{\zeta}(c_1, c_2)$ is contained in $\mathcal{M}_{\mathcal{C}}(c_1, c_2)$; thus, the right-hand side is contained in the left-hand side. Finally, let V be \mathcal{C} -stable but not \mathcal{F} -stable. Then, V sits in

$$0 \rightarrow \mathcal{O}_X(F) \rightarrow V \rightarrow \mathcal{O}_X(c_1 - F) \otimes I_Z \rightarrow 0,$$

where $(2F - c_1) \cdot L < 0 \leq (2F - c_1) \cdot L_0$ for $L \in \mathcal{C}$ and $L_0 \in \mathcal{F}$. Let $(2F - c_1) \equiv \zeta$. Then, ζ defines a wall of type (c_1, c_2) separating L and L_0 . Since \mathcal{F} is a face of \mathcal{C} , W^{ζ} must contain \mathcal{F} . Since $V \in E_{\zeta}(c_1, c_2)$, we conclude that the left-hand side is contained in the right-hand side. Hence, the equality holds.

Corollary 1.3.2. *Let \mathcal{C} be a chamber, and let \mathcal{F} be one of its faces. Then, $\mathcal{F} \subseteq \Delta_{\mathcal{C}}$ if and only if $E_{\zeta}(c_1, c_2)$ is empty for any ζ where ζ satisfies $\zeta \cdot L < 0$ for some $L \in \mathcal{C}$, and defines the wall containing \mathcal{F} .*

This gives a necessary and sufficient condition for a face of a chamber to be contained in the equivalence class determined by the chamber. It sharpens Corollary 2.2.2 in Chapter I. We will study the nonemptiness of $E_{\zeta}(c_1, c_2)$ in §2.

Theorem 1.3.3. *Let $\mathcal{C}, \mathcal{C}_1$ be chambers sharing a common wall. Then, as sets,*

$$\mathcal{M}_{\mathcal{C}}(c_1, c_2) = \left(\mathcal{M}_{\mathcal{C}_1}(c_1, c_2) - \coprod_{\zeta} E_{(-\zeta)}(c_1, c_2) \right) \coprod \left(\coprod_{\zeta} E_{\zeta}(c_1, c_2) \right),$$

where ζ satisfies $\zeta \cdot L < 0$ for some $L \in \mathcal{C}$, and runs over all numerical equivalence classes which define the common wall.

Proof. Follows immediately from Proposition 1.3.1.

By Proposition 1.3.1, $\coprod_{\zeta} E_{\zeta}(c_1, c_2)$ is the subset of $\mathcal{M}_{\mathcal{C}}(c_1, c_2)$ representing all sheaves which are not \mathcal{F} -stable. Since stability is an open property (see [20]), $\coprod_{\zeta} E_{\zeta}(c_1, c_2)$ is closed in $\mathcal{M}_{\mathcal{C}}(c_1, c_2)$. In general,

$E_\zeta(c_1, c_2)$ is neither open nor closed. A subset is defined to be *locally closed* if it is open in its closure, and *constructible* if it is a finite disjoint union of locally closed subsets (see [11]).

Proposition 1.3.4. *Let \mathcal{C} be a chamber such that $W^\zeta \cap \text{Closure}(\mathcal{C})$ is nonempty and $L \cdot \zeta < 0$ for $L \in \mathcal{C}$. Then, $E_\zeta(c_1, c_2)$ is a constructible subset in $\mathcal{M}_{\mathcal{C}}(c_1, c_2)$.*

Proof. Using the standard construction in [12], one sees that $E_\zeta(c_1, c_2)$ is quasiprojective; moreover, its scheme structure coincides with the induced one from $\mathcal{M}_{\mathcal{C}}(c_1, c_2)$. Therefore, $E_\zeta(c_1, c_2)$ is a constructible subset in the moduli space $\mathcal{M}_{\mathcal{C}}(c_1, c_2)$ by a theorem of Chevalley (see [3] and [11]). q.e.d.

Finally, we make some distinctions among different irreducible components.

Definition 1.3.5. Let $\mathcal{M}_L(c_1, c_2)$ be a nonempty moduli space. An irreducible component is defined to be *nontrivial* if it contains an open subset in which no sheaf is stable with respect to some polarization L' .

Proposition 1.3.6. *Let \mathcal{M} be a nontrivial irreducible component in $\mathcal{M}_L(c_1, c_2)$. Then, an open subset of \mathcal{M} is contained in some $E_\zeta(c_1, c_2)$ where ζ defines a nonempty wall of type (c_1, c_2) with $\zeta \cdot L < 0$.*

Proof. Let L' be a polarization with respect to which sheaves in an open subset U of \mathcal{M} are not stable. By Proposition 2.1.6 in Chapter I, the set of walls of type (c_1, c_2) is locally finite. Therefore, there are finitely many walls of type (c_1, c_2) separating L' and L , and there are finitely many ζ 's representing each of these walls and satisfying $\zeta \cdot L < 0 \leq \zeta \cdot L'$. Since each sheaf in U must be contained in some $E_\zeta(c_1, c_2)$, an open subset of U is contained in some $E_\zeta(c_1, c_2)$.

Remark 1.3.7. Let L' be as in the above proof. By Proposition 1.2.5, the ζ in Proposition 1.3.6 is unique if we require that $\zeta \cdot L'$ is nonnegative.

2. Nonemptiness of $E_\zeta(c_1, c_2)$

2.1. Classical techniques. Let Z stand for a locally complete intersection 0-cycle on a surface X .

Proposition 2.1.1 (see [10] and [2]). *Suppose Z consists of n distinct points $\{p_1, \dots, p_n\}$. Then, a locally free extension of $\mathcal{O}_X(L') \otimes I_Z$ by $\mathcal{O}_X(L)$ exists if and only if every section of $\mathcal{O}_X(L' - L + K_X)$ which vanishes at all but one of the p_i vanishes at the remaining point as well where K_X is the canonical divisor of X .*

Corollary 2.1.2. *Let $l > 2h^0(X, \mathcal{O}_X(L' - L + K_X))$. Then, there exists Z with length l such that a locally free extension of $\mathcal{O}_X(L') \otimes I_Z$ by $\mathcal{O}_X(L)$ exists.*

2.2. Nonemptiness of $E_\zeta(c_1, c_2)$.

Definition 2.2.1. For any divisor F with $(2F - c_1) \equiv \zeta$, we define $E_F(c_1, c_2)$ to be the set of those sheaves in $E_\zeta(c_1, c_2)$ for which $\mathcal{O}_X(F)$ is a subsheaf.

Remark 2.2.2. Let F be as in the above definition. Then $E_{(c_1-F)}(c_1, c_2)$ is the set of those sheaves in $E_{(-\zeta)}(c_1, c_2)$ for which $\mathcal{O}_X(c_1 - F)$ is a subsheaf.

In the following, we fix F such that $(2F - c_1) \equiv \zeta$, and study the nonemptiness of $E_F(c_1, c_2)$ and $E_{(c_1-F)}(c_1, c_2)$ since it implies the nonemptiness of $E_\zeta(c_1, c_2)$ and $E_{(-\zeta)}(c_1, c_2)$. First of all, we impose no condition on the wall W^ζ .

Lemma 2.2.3. *If the linear system $|K_X + (2F - c_1)|$ is empty, then $E_F(c_1, c_2)$ or $E_{(c_1-F)}(c_1, c_2)$ is nonempty when $c_2 > c_1^2 + 2\chi(\mathcal{O}_X)/4$.*

Proof. If $c_2 + (\zeta^2 - c_1^2)/4 > 0$, then $E_{(c_1-F)}(c_1, c_2)$ is nonempty by Corollary 2.1.2. If $c_2 + (\zeta^2 - c_1^2)/4 = 0$ (i.e., $\zeta^2 = -(4c_2 - c_1^2)$), then $E_{(c_1-F)}(c_1, c_2)$ is nonempty unless $\text{Ext}^1(\mathcal{O}_X(F), \Omega_X(c_1 - F)) = 0$, that is, $H^1(X, \Omega_X(c_1 - 2F)) = 0$.

Assume both $\zeta^2 = -(4c_2 - c_1^2)$ and $H^1(X, \mathcal{O}_X(c_1 - 2F)) = 0$. Note that $h^2(X, \mathcal{O}_X(c_1 - 2F)) = h^0(X, \mathcal{O}_X(K_X + 2F - c_1)) = 0$ by assumption, and that $h^0(X, \mathcal{O}_X(c_1 - 2F)) = 0$ for $(c_1 - 2F) \equiv -\zeta$. By the Riemann-Roch formula, we have $\chi(\mathcal{O}_X) + \frac{1}{2} \cdot (c_1 - 2F)[(c_1 - 2F) - K_X] = 0$. Thus, $K_X \cdot \zeta = -2\chi(\mathcal{O}_X) - \zeta^2$. We now consider $E_F(c_1, c_2)$. Since $h^0(X, \mathcal{O}_X(2F - c_1)) = 0$, we have

$$\begin{aligned} h^1(X, \mathcal{O}_X(2F - c_1)) &\geq -\chi(\mathcal{O}_X) - \frac{1}{2}(2F - c_1)(2F - c_1)[(2F - c_1) - K_X] \\ &= (4c_2 - c_1^2) - 2\chi(\mathcal{O}_X). \end{aligned}$$

Therefore, if $c_2 > (c_1^2 + 2\chi(\mathcal{O}_X))/4$ then $h^1(X, \mathcal{O}_X(2F - c_1)) > 0$; thus, $E_F(c_1, c_2)$ is nonempty since $\text{Ext}^1(\mathcal{O}_X(c_1 - F), \mathcal{O}_X(F))$ is nontrivial.

Lemma 2.2.4. *If $|K_X + (2F - c_1)|$ and $|K_X - (2F - c_1)|$ are nonempty, then $E_F(c_1, c_2)$ and $E_{(c_1-F)}(c_1, c_2)$ are nonempty if $c_2 > c(X, c_1)$ where $c(X, c_1)$ is a constant depending on X and c_1 .*

Proof. Fix a polarization $H \in \text{Num}(X)$ with $H^2 = 1$. Let $\{e_1, \dots, e_r\}$ be a basis for the space $\{x \in \text{Num}(X) | x \cdot H = 0\}$ with $e_i \cdot e_j = -\delta_{ij}$. Choose a positive number a such that $aH + e_i$ is a polarization for any

i. Now, $0 \leq H \cdot [K_X \pm (2F - c_1)]$ implies that $|H \cdot \zeta| \leq H \cdot K_X$. Put $\zeta = (H \cdot \zeta)H + \sum_i \zeta_i e_i$. Then,

$$\begin{aligned} 0 &\leq (aH + e_i) \cdot [K_X \pm (2F - c_1)] = a(H \cdot K_X) \pm a(H \cdot \zeta) + (K_X \cdot e_i) \mp \zeta_i \\ &\leq 2a(H \cdot K_X) + |(K_X \cdot e_i)| \mp \zeta_i. \end{aligned}$$

Thus, $|\zeta_i| \leq 2a(H \cdot K_X) + b$ for any i where we have put

$$b = \max\{|(K_X \cdot e_i)| \mid i = 1, \dots, r\}.$$

So, $|\zeta^2| \leq (H \cdot \zeta)^2 + \sum_i |\zeta_i|^2 \leq b_1$ where b_1 is a constant depending on X (and H). Since $|K_X + (2F - c_1)|$ and $|K_X - (2F - c_1)|$ are nonempty, $h^0(X, \mathcal{O}_X(K_X \pm (2F - c_1)))$ is not greater than $h^0(X, \mathcal{O}_X(2K_X))$. Put

$$c(X, c_1) = 2h^0(X, \mathcal{O}_X(2K_X)) + (b_1 + c_1^2)/4.$$

Assume $c_2 > c(X, c_1)$. Then

$$c_2 + (\zeta^2 - c_1^2)/4 > 2h^0(X, \mathcal{O}_X(K_X \pm (2F - c_1))).$$

By Corollary 2.1.2, both $E_F(c_1, c_2)$ and $E_{(c_1-F)}(c_1, c_2)$ are nonempty.

Theorem 2.2.5. *For any numerical equivalence class ζ which defines a nonempty wall of type (c_1, c_2) , $E_\zeta(c_1, c_2)$ or $E_{(-\zeta)}(c_1, c_2)$ is nonempty if $c_1 > c(X, c_1)$ where $c(X, c_1)$ is a constant depending on X and c_1 .*

Proof. Let F be any divisor with $(2F - c_1) \equiv \zeta$. There are two cases:

- (i) $|K_X + (2F - c_1)|$ or $|K_X - (2F - c_1)|$ is empty;
- (ii) both $|K_X + (2F - c_1)|$ and $|K_X - (2F - c_1)|$ are nonempty.

For case (i), we apply Lemma 2.2.3. For case (ii), we apply Lemma 2.2.4.

Corollary 2.2.6. *Let \mathcal{E}_1 and \mathcal{E}_2 be two chambers of type (c_1, c_2) sharing a common face which is part of the wall W . Then for any $L_0 \in W$, $L_0 \notin \mathcal{E}_{\mathcal{E}_1}$ or $L_0 \notin \mathcal{E}_{\mathcal{E}_2}$ if $c_2 > c(X, c_1)$ where $c(X, c_1)$ is a constant depending on X and c_1 .*

Proof. Let $c(X, c_1)$ be the same as in Theorem 2.2.5. Assume W is represented by ζ . If $c_2 > c(X, c_1)$, then $E_\zeta(c_1, c_2)$ or $E_{(-\zeta)}(c_1, c_2)$ is nonempty by Theorem 2.2.5. We may assume that $E_\zeta(c_1, c_2)$ is nonempty and that $\zeta \cdot L < 0$ for some $L \in \mathcal{E}_1$. Let $V \in E_\zeta(c_1, c_2)$. By Theorem 1.2.3, V is \mathcal{E}_1 -stable but strictly L_0 -semistable. By the definition of equivalence classes, we conclude that $L_0 \notin \mathcal{E}_{\mathcal{E}_1}$.

Remark 2.2.7. By the results in [21] and [22], the condition that $c_2 \gg 0$ in Theorem 2.2.5 cannot be weakened, and Theorem 2.2.5 is the sharpest.

Next, let \mathcal{B} be a compact subset in the Kähler cone \mathbf{C}_X . We now investigate the nonemptiness of $E_\zeta(c_1, c_2)$ where the wall W^ζ intersects

with \mathcal{B} . Let H and e_1, \dots, e_r be as in the proof of Lemma 2.2.4. Since \mathcal{B} is compact, we can choose a positive integer a satisfying the following properties:

- (P1) $(aL \pm H) \in C_X$ for any L in \mathcal{B} ;
- (P2) $(aH \pm e_i) \in C_X$ for any i ;
- (P3) $(aH - K_X)$ is strictly effective and aH is very ample.

Assume $W^\zeta \cap \mathcal{B}$ is nonempty, and $(2F - c_1) \equiv \zeta$. We separate the case where $|K_X - (2F - c_1)|$ is nonempty and the case where $|K_X - (2F - c_1)|$ is empty.

Lemma 2.2.8. *If $|K_X - (2F - c_1)|$ is nonempty, then $E_F(c_1, c_2)$ is nonempty when $c_2 > c(X, c_1, \mathcal{B})$ where $c(X, c_1, \mathcal{B})$ is a constant depending on X, c_1 and \mathcal{B} .*

Proof. First of all, we show that $|\zeta^2|$ is bounded above by some constant depending on X and \mathcal{B} . Put $\zeta = (H \cdot \zeta)H + \sum_i \zeta_i e_i$. Let $L \in \mathcal{B} \cap W^\zeta$. We have $0 < [K_X - (2F - c_1)] \cdot (aL \pm H) = (K_X - \zeta) \cdot (aL \pm H)$, so $|H \cdot \zeta| < a|K_X \cdot L| + |K_X \cdot H|$ for $L \cdot \zeta = 0$. Now, $0 < [K_X - (2F - c_1)] \cdot (aH \pm e_i)$ implies that $|\zeta_i| < a^2|K_X \cdot L| + 2a|K_X \cdot H| + |K_X \cdot e_i|$. Thus, $|\zeta^2| \leq |H \cdot \zeta|^2 + \sum_i |\zeta_i|^2 < b_1$ where b_1 is a constant depending on X and \mathcal{B} .

Next, we show that $h = h^0(X, \mathcal{O}_X(K_X - (2F - c_1)))$ is also bounded above. Let $E \in |aH|$ be a smooth curve, and let g_E be its genus. Then we have

$$0 \rightarrow \mathcal{O}_X(K_X - (2F - c_1) - E) \rightarrow \mathcal{O}_X(K_X - (2F - c_1)) \rightarrow \mathcal{O}_E(D) \rightarrow 0,$$

where $\mathcal{O}_E(D)$ is the restriction of $\mathcal{O}_X(K_X - (2F - c_1))$ on E . Since $(aH - K_X)$ is strictly effective, $[K_X - (2F - c_1) - E] \cdot L = (K_X - aH) \cdot L < 0$, so $(K_X - (2F - c_1) - E)$ can never be effective. Thus, $h \leq h^0(E, \mathcal{O}_E(D))$. Note that

$$\deg(D) \leq a|K_X \cdot H| + a|\zeta \cdot H| < a^2|K_X \cdot L| + 2a|K_X \cdot H|.$$

If $h^1(E, \mathcal{O}_E(D)) = 0$, then

$$h^0(E, \mathcal{O}_E(D)) = \deg(D) + 1 - g_E \leq (a^2|K_X \cdot L| + 2a|K_X \cdot H| + 1).$$

If $h^1(E, \mathcal{O}_E(D)) > 0$, then by the Clifford's Theorem [11], $h^0(E, \mathcal{O}_E(D)) \leq 1 + \deg(D)/2 \leq (a^2|K_X \cdot L| + 2a|K_X \cdot H| + 1)$. From either case, we conclude $h \leq b_2$ where $b_2 = (a^2|K_X \cdot L| + 2a|K_X \cdot H| + 1)$ is a constant depending on X and \mathcal{B} . Put $c(X, c_1, \mathcal{B}) = 2b_2 + (b_1 + c_1^2)/4$.

If $c_2 > c(X, c_1, \mathcal{B})$, then

$$c_2 + (\zeta^2 - c_1^2)/4 > 2h^0(X, \mathcal{O}_X(K_X - (2F - c_1))).$$

By Corollary 2.1.2, we conclude that $E_F(c_1, c_2)$ is nonempty.

We now move to the case where $|K_X - (2F - c_1)|$ is empty. Let $\overline{O\mathcal{B}}$ be the cone with vertex O (the trivial element in $\text{Num}(X) \otimes \mathbf{R}$), and spanned by elements in \mathcal{B} . Any wall intersecting with \mathcal{B} must contain a ray in $\overline{O\mathcal{B}}$. Using this observation and the assumption that \mathcal{B} is compact, we can choose another compact subset \mathcal{B}' in C_X large enough such that any wall intersecting with \mathcal{B} cuts \mathcal{B}' into two parts, and that each part contains the image of some very ample divisor.

Lemma 2.2.9. *If $|K_X - (2F - c_1)|$ is empty, then $E_F(c_1, c_2)$ is nonempty when $c_2 > c(X, c_1, \mathcal{B})$ where $c(X, c_1, \mathcal{B})$ is a constant depending on X, c_1 and \mathcal{B} .*

Proof. Clearly, $E_F(c_1, c_2)$ is nonempty unless $\zeta^2 = -(4c_2 - c_1^2)$ and $H^1(X, \mathcal{O}_X(2F - c_1)) = 0$. Assume $\zeta^2 = -(4c_2 - c_1^2)$ and

$$H^1(X, \mathcal{O}_X(2F - c_1)) = 0.$$

We need to show that c_2 is bounded above by some constant depending on X, c_1 and \mathcal{B} .

Since $\chi(\mathcal{O}_X(2F - c_1)) = 0$, by the Riemann-Roch formula, we obtain $\zeta \cdot K_X = 2\chi(\mathcal{O}_X) + \zeta^2$, so $\zeta \cdot K_X = 2\chi(\mathcal{O}_X) - (4c_2 - c_1^2)$. Since W^ζ intersects with \mathcal{B} , from the assumption on \mathcal{B}' , there exists a very ample divisor H' such that its image in $\text{Num}(X) \otimes \mathbf{R}$ is contained in \mathcal{B}' and that $\zeta \cdot H' < 0$. Note that $2H'$ is very ample. Choose a smooth curve $E \in |2H'|$, and let g_E be its genus. Consider

$$0 \rightarrow \mathcal{O}_X(K_X - (2F - c_1)) \rightarrow \mathcal{O}_X(E + K_X - (2F - c_1)) \rightarrow \mathcal{O}_E(D) \rightarrow 0,$$

where $\mathcal{O}_E(D)$ is the restriction of $\mathcal{O}_X(E + K_X - (2F - c_1))$ on E . Since both $h^0(X, \mathcal{O}_X(K_X - (2F - c_1)))$ and $h^1(X, \mathcal{O}_X(K_X - (2F - c_1)))$ are 0, $h^0(X, \mathcal{O}_X(E + K_X - (2F - c_1))) = h^0(E, \mathcal{O}_E(D))$. We have $\deg(D) = (2g_E - 2) - 2(\zeta \cdot H')$. Thus, $h^0(E, \mathcal{O}_E(D)) > 0$ since $(\zeta \cdot H') < 0$, so $|(E + K_X) - (2F - c_1)|$ is nonempty. As in the proof of Lemma 2.2.8, we can show that ζ^2 is bounded above by some constant depending on X, \mathcal{B} (and \mathcal{B}'). Therefore, $|c_2| = |(c_1^2 - \zeta^2)|/4$ is bounded above by some constant depending on X, c_1 and \mathcal{B} .

Theorem 2.2.10. *Let \mathcal{B} be a compact subset in the Kähler cone C_X . For any numerical equivalence class ζ with $W^\zeta \cap \mathcal{B}$ to be nonempty, $E_\zeta(c_1, c_2)$ is nonempty when $c_2 > c(X, c_1, \mathcal{B})$ where $c(X, c_1, \mathcal{B})$ is a constant depending on X, c_1 and \mathcal{B} .*

Proof. Let F be any divisor with $(2F - c_1) \equiv \zeta$. If $|K_X - (2F - c_1)|$ is nonempty, we apply Lemma 2.2.8. If $|K_X - (2F - c_1)|$ is empty, we use Lemma 2.2.9.

Corollary 2.2.11. *Let \mathcal{B} be a compact subset in the Kähler cone \mathcal{C}_X , let W be a wall of type (c_1, c_2) , and let \mathcal{C} be a chamber. Assume both $W \cap \mathcal{B}$ and $W \cap \text{Closure}(\mathcal{C})$ are nonempty. Then $W \cap \mathcal{E}_{\mathcal{C}}$ is empty if $c_2 > c(X, c_1, \mathcal{B})$ where $c(X, c_1, \mathcal{B})$ is a constant depending on X, c_1 and \mathcal{B} .*

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